Lecture 12

Note Title 5/26/2009

Ekman Layers

So far we have neglected the effects of friction in the dynamics of the flows we have been considering. In this lecture we will show that although frictional forces are quite weak in most of the water column, in boundary layers near the surface and bottom of the ocean friction cannot be neglected and generates flows that have an affect on the dynamics of the whole water column.

The equations of motion including friction are:

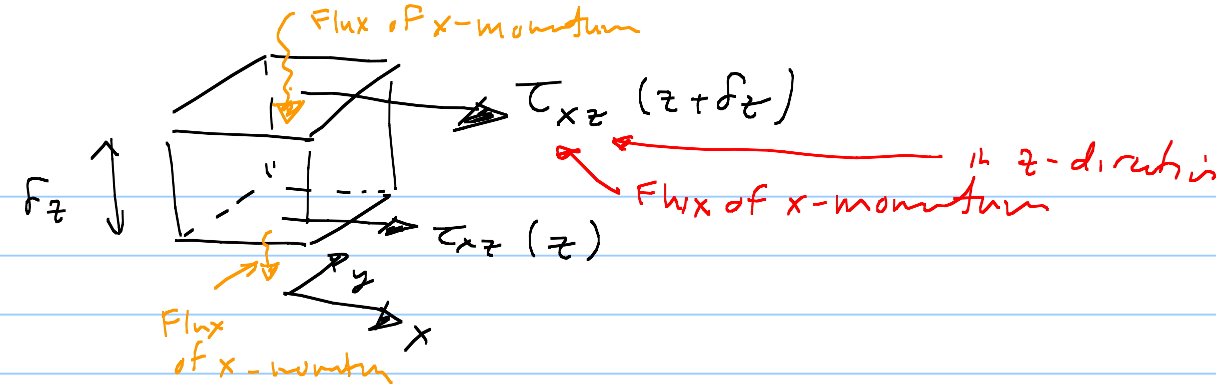
$$\frac{D\vec{u}}{Dt} + f\hat{k}\times\vec{u} = -\frac{1}{\rho_0}\nabla p + b\hat{k} + \nabla\cdot\left(\frac{\overleftrightarrow{\tau}}{\rho_0}\right)$$

$\overleftrightarrow{\tau}$ is the stress tensor

stress = − Flux of momentum

vector   vector quantity

$$\nabla\cdot\left(\frac{\overleftrightarrow{\tau}}{\rho_0}\right) = \underbrace{-\nabla\cdot\left(\frac{\text{Flux of momentum}}{\rho_0}\right)}_{\text{Convergence of a momentum flux}}$$



→ Convergence of momentum flux → accelerate the fluid in the x-direction

In the ocean, you will recall that the vertical variations of the flows that we have considered are typically much larger than the lateral variations of the flow, i.e. in terms of the vertical (H) and lateral (L) lengthscales of the flow:

$$\frac{H}{L} \ll 1$$

This implies that most of the variation in the stress is in the vertical rather than the horizontal, i.e.:

$$\nabla \cdot \left(\frac{\overleftrightarrow{\tau}}{\rho_0} \right) \simeq \frac{\partial}{\partial z} \left(\frac{\overleftrightarrow{\tau_V}}{\rho_0} \right)$$

vertical component of momentum flux

However, superimposed on the "large" scale flows that we are interested in modeling are small scale turbulent motions that can flux momentum. The flux of momentum associated with these turbulent motions can be quantified using Reynolds averaging:

$$\frac{\overleftrightarrow{\tau}_v}{\rho_0} = -\overbrace{\overline{\vec{u}'w'}}^{\text{turbulent momentum flux}}_{\text{REYNOLDS STRESS}} + \overbrace{\nu \frac{\partial \vec{\bar{u}}}{\partial z}}^{\text{momentum flux associated with molecular viscosity}}$$

Where primes denote a quantity associated with the turbulent motions and an overline denotes an average over the turbulent motions (could be a spatial, temporal, ensemble average).

The Reynolds stress is commonly parameterized in terms of an EDDY VISCOSITY:

$$-\overline{\vec{u}'w'} = \nu_{ed} \frac{\partial \vec{\bar{u}}}{\partial z}$$

Where it is assumed that the turbulent motions mix momentum similar to molecular viscosity but with a much larger magnitude. Observational estimates of ν_{ed}

$$10^{-5} \frac{m^2}{s} < \nu_{ed} < 0.01 \frac{m^2}{s}$$

Whereas the molecular viscosity of water is $\nu = 10^{-6}\ m^2/s$. Note that the eddy diffusivity/viscosity is spatially variable since it depends on the properties of the turbulence.

Having said that lets assume that it is constant for the time being. Neglecting the contribution to the stress by molecular viscosity the equations of motion become:

$$\frac{D\vec{u}}{Dt} + f\hat{k}\times\vec{u} = -\frac{1}{\rho_0}\nabla p + b\hat{k} + \nu_{ed}\frac{\partial^2 \vec{u}}{\partial z^2}$$

When the overtines have been dropped and it is assumed that the flow is associated with the "large" scale, mean flow.

Lets nondimensionalize the equations

$$(u,v) = U(\hat{u},\hat{v}) \qquad w = U\frac{H}{L}\hat{w}$$

$$p = \rho_0 f L U \hat{p} \qquad z = H\hat{z} \quad (x,y) = L(\hat{x},\hat{y})$$

$$t = T\hat{t}$$

Lets consider a homogeneous fluid i.e. $\rho = \rho_0 \rightarrow b = 0$.

Subsituting these forms into the equations of motion yields

$$Ro_t \frac{\partial \hat{u}}{\partial \hat{t}} + Ro\,\vec{\hat{u}}\cdot\nabla\hat{u} - \hat{v} = -\frac{\partial \hat{p}}{\partial \hat{x}} + Ek\frac{\partial^2 \hat{u}}{\partial \hat{z}^2}$$

$$Ro_t \frac{\partial \hat{v}}{\partial t} + Ro\,\vec{\hat{u}}\cdot\nabla\hat{v} + \hat{u} = -\frac{\partial \hat{p}}{\partial \hat{y}} + Ek\frac{\partial^2 \hat{v}}{\partial \hat{z}^2}$$

$$Ro_t \Gamma^2 \frac{\partial \hat{w}}{\partial \hat{t}} + Ro \Gamma^2 \vec{\hat{u}} \cdot \nabla \hat{w} = -\frac{\partial \hat{p}}{\partial \hat{z}} + \Gamma^2 Ek \frac{\partial^2 \hat{w}}{\partial \hat{z}^2}$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} + \frac{\partial \hat{w}}{\partial \hat{z}} = 0$$

where the non-dimensional parameters are

$Ro_t = \frac{1}{fT}$ (Ratio of inertial period to timescale of flow)

$Ro = \frac{U}{fL}$ (Ratio of advective terms to Coriolis force)

$\Gamma = H/L$ (aspect ratio of flow)

$Ek = \frac{\nu_{ed}}{fH^2}$ (ratio of frictional force to Coriolis force)

EKMAN NUMBER

How big is the Ekman number in the ocean?

To estimate the Ekman number let's use the following values:

$\nu_{ed} = 10^{-3}\ m^2/s$ (on the large side)

$H = 1000\ m$

$f = 10^{-4}\ s^{-1}$ mid-latitude

$$Ek = \frac{10^{-3}}{10^{-4}\,10^{6}} = 10^{-5} \ll 1$$

This suggests that we can neglect friction from the equations of motion. Assuming small Ro, Ro_t, Γ the equations of motion would become

$$-\hat{v} = -\frac{\partial \hat{p}}{\partial x} \qquad + \hat{u} = -\frac{\partial \hat{p}}{\partial \hat{y}}$$

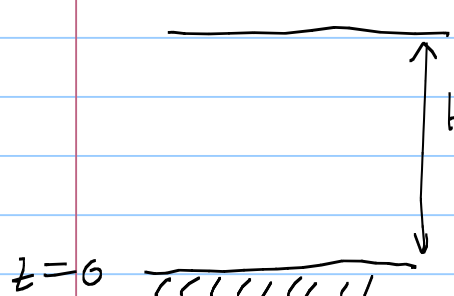
$$0 = -\frac{\partial \hat{p}}{\partial z}$$

Thus in this limit of $Ro, Ro_t, Ek, \Gamma \ll 1$ the flow is geostrophic (i.e. in dimensional units:

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

$$(u, v) = (u_g, v_g)$$

But what if we had a bottom to the ocean and we impose a no-slip boundary condition, i.e.



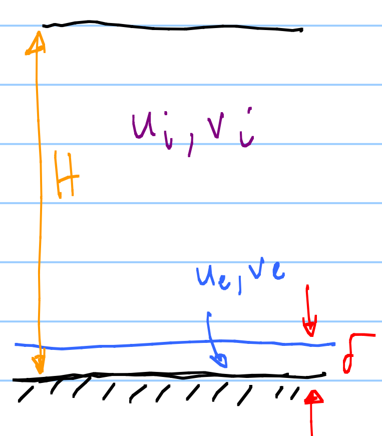
$$u = v = 0 \text{ at } z = 0$$

Does the geostrophic flow satisfy this no-slip boundary condition?

For this barotropic flow where $\frac{\partial u_g}{\partial z} = \frac{\partial v_g}{\partial z} = 0$ for non-zero geostrophic flow, the geostrophic flow cannot satisfy the bottom boundary condition. Thus the solutions to the equations of motion when $Ek = 0$ do not satisfy the boundary conditions, thus we cannot fully neglect the frictional terms in the equations of motion if we wish to satisfy the bottom boundary condition.

What happens? Well basically a boundary layer develops near the bottom that is very thin and where the frictional forces are as strong as the Coriolis force. How thin is this boundary layer?

Lets split the flow into two parts



In the boundary layer the flow u_e, v_e has a vertical length scale δ that is much smaller than the total thickness of the water column. How thin does δ have to be for friction to be as important as the Coriolis force?

$$\left|\frac{\nu_{ed}\,\partial^2 u_e/\partial z^2}{-f v_e}\right| \sim \frac{\nu_{ed}}{\delta^2}\frac{U}{Uf} \sim 1$$

$$\Rightarrow \delta \sim \sqrt{\frac{\nu_{ed}}{f}}$$

While in the interior,

$$\left|\frac{\nu_{ed}\,\partial^2 u_i/\partial z^2}{-f v_i}\right| \sim \frac{\nu_{ed}}{H^2}\frac{U}{Uf} = Ek \ll 1$$

So by splitting the flow into interior and boundary layer components, where it is assumed that the boundary layer component of the flow decays to zero outside of the boundary layer

$$u_e, v_e \to 0 \quad z/\delta \gg 1$$

And denoting the interior as the region above the bottom where

$$z \gg \delta$$

Then $\quad u = u_i \quad v = v_i \quad$ for $z \gg \delta$

and $\quad u = u_i + u_e \quad v = v_i + v_e$ for $z < \delta$

Then the equations of motion in the interior are:

$$-f v_i = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad f u_i = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

i.e. the flow is geostrophic

and $$-f(v_i+v_e) = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu_e \frac{\partial^2 u_e}{\partial z^2}$$

$$+f(u_i+u_e) = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu_e \frac{\partial^2 v_e}{\partial z^2}$$

$z \sim \delta$
$z < \delta$

The z-momentum equation states:

$$0 = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \rightarrow$$ the pressure does not vary with height

Therefore in the boundary layer

$$-f v_i = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \qquad +f u_i = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$$

So that the mom equations in the boundary layer become:

$$-f v_e = \nu_e \frac{\partial^2 u_e}{\partial z^2} \qquad \text{A}$$

$$f u_e = \nu_e \frac{\partial^2 v_e}{\partial z^2} \qquad \text{B}$$

The no-slip boundary condition is

$$u = u_i + u_e = 0 \quad \text{at } z=0$$
$$v = v_i + v_e = 0$$

So if some interior geostrophic flow is specified, we can solve for the flow in the boundary layer that is required to satisfy the no slip b c!

$$u_e = -u_i \qquad v_e = -v_i \qquad \text{at } z=0$$

Lets do this:

A & B can be combined into a single equation by taking for example $\frac{\partial^2}{\partial z^2}$ A

$$\Rightarrow \quad \nu_e \frac{\partial^4 u_e}{\partial z^4} = -f \frac{\partial^2 v_e}{\partial z^2} = -\frac{f^2}{\nu_e} u_e$$

$$\frac{\partial^4 u_e}{\partial z^4} + \frac{f^2}{\nu_e^2} u_e = 0 \qquad \text{C}$$

This is an ODE with constant coefficients so we can solve it using a solution of the form:

$$u_e \sim e^{\lambda z}$$

λ complex number

Substituting this into C:

$$\lambda^4 + \frac{f^2}{\nu_e^2} = 0$$

$$\lambda^4 = -\frac{f^2}{\nu_e^2} \qquad \lambda^2 = \pm \underbrace{\sqrt{-1}}_{i} \frac{f}{\nu_e}$$

$$\lambda = \pm \left\{ \sqrt{-1}\sqrt{i}\sqrt{\frac{f}{\nu_e}},\ \sqrt{i}\sqrt{\frac{f}{\nu_e}} \right\}$$

$\Rightarrow$ Four roots

One of the boundary conditions on u_e is that $u_e \to 0$ as $z \gg \sqrt{\frac{\nu_e}{f}} = \delta$

Thus we must pick the root where soln decays w/ height
What is $\sqrt{i}$?

$$i = e^{i\pi/2} = \cancelto{0}{\cos(\pi/2)} + i \cancelto{1}{\sin(\pi/2)}$$

$$\sqrt{i} = e^{i\pi/4} = \cos(\pi/4) + i\sin(\pi/4)$$
$$= \frac{1}{\sqrt{2}}(1 + i)$$

$$i\sqrt{i} = \frac{1}{\sqrt{2}}(i - 1)$$

$$\lambda = \sqrt{\frac{f}{2\nu_e}} \left\{ (i-1), \underbrace{(1+i), (1-i)}, -(1+i) \right\}$$

can't have these roots because they increase w/ height

Note that $\delta_e = \sqrt{\frac{2\nu_e}{f}}$ is the characteristic lengthscale of the boundary layer and it known as the Ekman layer depth

So that the general solution for u_e is:

$$u_e = \mathrm{Re}\left\{ (A_r + iA_i)\, e^{-z/\delta_e} e^{iz/\delta_e} + (B_r + iB_i)\, e^{-z/\delta_e} e^{-iz/\delta_e} \right\}$$

which can be simplified to the form:

$$u_e = C e^{-z/\delta_e} \sin(z/\delta_e)$$
$$+ D e^{-z/\delta_e} \cos(z/\delta_e)$$

when C & D are real numbers.

We can now use the equation of motion in the boundary layer to solve for v_e:

$$v_e = -\frac{\nu_e}{f} \frac{\partial^2 u_e}{\partial z^2}$$

$$\frac{\partial u_e}{\partial z} = \frac{-C}{\delta_e} e^{-z/\delta_e} \sin(z/\delta_e) - \frac{D}{\delta_e} e^{-z/\delta_e} \cos(z/\delta_e)$$

$$+ \frac{C}{\delta_e} e^{-z/\delta_e} \cos(z/\delta_e) - \frac{D}{\delta_e} e^{-z/\delta_e} \sin(z/\delta_e)$$

$$= -\frac{(C+D)}{\delta_e} e^{-z/\delta_e} \sin\left(\frac{z}{\delta_e}\right) + \frac{(C-D)}{\delta_e} e^{-z/\delta_e} \cos\left(\frac{z}{\delta_e}\right)$$

$$\frac{\partial^2 u_e}{\partial z^2} = + \frac{(C+D)}{\delta_e^2} e^{-z/\delta_e} \sin(z/\delta_e) - \frac{(C+D)}{\delta_e^2} e^{-z/\delta_e} \cos\left(\frac{z}{\delta_e}\right)$$

$$- \frac{(C-D)}{\delta_e^2} e^{-z/\delta_e} \cos\left(\frac{z}{\delta_e}\right) - \frac{(C-D)}{\delta_e^2} e^{-z/\delta_e} \sin(z/\delta_e)$$

$$= \frac{2D}{\delta_e^2} e^{-z/\delta_e} \sin\left(\frac{z}{\delta_e}\right) - \frac{2C}{\delta_e^2} e^{-z/\delta_e} \cos\left(\frac{z}{\delta_e}\right)$$

$$v_e = -D e^{-z/\delta_e} \sin\left(\frac{z}{\delta_e}\right) + C e^{-z/\delta_e} \cos\left(\frac{z}{\delta_e}\right)$$

To find the coefficients C & D we use the bottom boundary condition:

$$u_e = -u_i \qquad v_e = -v_i \qquad \text{at } z=0$$

$$\Rightarrow \quad -u_i = D \qquad -v_i = C$$

$$\boxed{\begin{aligned} u_e &= -v_i e^{-z/\delta_e} \sin\left(\frac{z}{\delta_e}\right) - u_i e^{-z/\delta_e} \cos\left(\frac{z}{\delta_e}\right) \\ v_e &= u_i e^{-z/\delta_e} \sin\left(\frac{z}{\delta_e}\right) - v_i e^{-z/\delta_e} \cos\left(\frac{z}{\delta_e}\right) \end{aligned}}$$

This is the so called Ekman spiral solution because the flow in this Ekman layer spirals with depth.

Show PPT

You will notice that the flow in the Ekman layer spirals to the left of the interior geostrophic flow.

Why is this?

Consider an interior flow that is purely in the zonal direction:

This flow is associated with a pressure gradient force that is directed to the north, in accordance with the force balance in the interior:

$$+fu_i = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} > 0$$

Now in the Ekman layer, the pressure gradient force is not balanced by the Coriolis force, i.e.

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial y} - f(u_i + u_e) \neq 0$$

Because the Ekman flow u_e reduces the total zonal velocity so as to satisfy the no-slip boundary condition

show ppt

Consequently the PGF is not completely balanced so that

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial y} - f(u_i + u_e) > 0 \quad \text{in the Ekman layer}$$

This imbalance would accelerate a flow down the pressure gradient, i.e. to the north, but this tendency

For the PGF to accelerate a northward flow is counteracted by friction i.e.

$$-\frac{1}{\rho_0}\frac{\partial p}{\partial y} - f\left(u_i + u_e\right) + \nu_e \frac{\partial^2 v_e}{\partial z^2} = 0$$

so that a steady balance is achieved.

Nonetheless the PGF drives a flow directed from high to low pressure in the Ekman layer and this is what causes the leftward veering of the Ekman flow.